

The stochastic wave equation in high dimensions: Malliavin differentiability and absolute continuity

by

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Abstract We consider the class of non-linear stochastic partial differential equations studied in [3]. Equivalent formulations using integration with respect to a cylindrical Brownian motion and also the Skorohod integral are established. It is proved that the random field solution to these equations at any fixed point $(t, x) \in [0, T] \times \mathbb{R}^d$ is differentiable in the Malliavin sense. For this, an extension of the integration theory in [3] to Hilbert-valued integrands is developed, and commutation formulae of the Malliavin derivative and stochastic and pathwise integrals are proved. In the particular case of equations with additive noise, we establish the existence of density for the law of the solution at $(t, x) \in [0, T] \times \mathbb{R}^d$.

Keywords: stochastic integration; stochastic partial differential equations; stochastic wave equation; Malliavin calculus; densities.

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1 Introduction

In this article, we consider stochastic partial differential equations (SPDEs) of the type

$$\begin{aligned} Lu(t, x) &= \sigma(u(t, x))\dot{F}(t, x) + b(u(t, x)), \quad (t, x) \in]0, T] \times \mathbb{R}^d, \\ u(0, x) &= \frac{\partial}{\partial t}u(0, x) = 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where L is a second order differential operator, σ and b are real functions, \dot{F} is the formal derivative of a Gaussian stochastic process and $d \in \mathbb{N}$. The setting applies in particular to the wave operator

$$\frac{\partial^2}{\partial t^2} - \Delta_d,$$

where Δ_d denotes the Laplacian in dimension d .

The main objective is to establish the differentiability in the Malliavin sense of the solution to the Cauchy problem (1.1), with a suitable formulation. As a by-product, the existence of density for a class of equations that include a stochastic wave equation in spatial dimension $d \geq 4$ with additive noise will be proved.

We give a rigorous meaning to (1.1) using the stochastic integration theory developed in [3], which extends the setting of [24] and [5]. More precisely, let Λ be the fundamental solution associated with the operator L . We consider the mild form of the equation (1.1):

$$\begin{aligned} u(t, x) &= \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \sigma(u(s, z)) M(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) b(u(s, z)) dz ds, \end{aligned} \tag{1.2}$$

where M denotes the martingale measure derived from the Gaussian process F . More specifications on F and on the stochastic and pathwise integrals used in (1.2) are given in Section 2. Under suitable conditions on Λ , and for Lipschitz continuous coefficients σ and b , [3, Theorem 4.2] establishes the existence of a random field solution to (1.2). Thus, it makes sense to fix $(t, x) \in]0, T] \times \mathbb{R}^d$ and to study the probability law of the real-valued random variable $u(t, x)$. A preliminary step consists of proving the Malliavin differentiability of $u(t, x)$.

Malliavin differentiability and existence and properties of the density have been studied in particular cases of (1.2), like the stochastic heat equation with $d \in \mathbb{N}$ and the stochastic wave equation with $d \in \{1, 2, 3\}$. We refer the reader to [1], [10], [11], [12], [13], [15], [16], [18], [19], [21], [20] for a sample of results. However, to the best of our knowledge, similar problems for the stochastic wave equation in dimension $d \geq 4$ have not been yet considered. The main difficulty stems from the non-smoothness properties of the distribution Λ . The results of this paper apply to this particular example.

Next, we describe the content of the article. Section 2 gathers the preliminary notions and results underpinning Equation (1.1). In particular, the governing noise F is described and the construction of the integrals in (1.2) sketched. Along with this, we prove a new result (see Lemma 2.1) that will be used later on in Section 4 to give a formulation of the stochastic integral in [3] in terms of a sequence of independent standard Brownian motions.

In Section 3 an extension of Conus-Dalang's stochastic and pathwise integrals to Hilbert-valued stochastic processes is developed. With this, we extend the results proved in [18] and provide the theoretical background for the study of the Malliavin derivative of the solution of (1.2).

Section 4 contains some complements to the Conus-Dalang's stochastic integral. For a relevant class of integrands, we prove that its divergence operator (in Malliavin sense) coincides with that integral and also with an Itô stochastic integral with respect to a sequence of independent standard Brownian motions. Actually, the latter is nothing but the stochastic integral with respect to a cylindrical Brownian motion, as in the setting of [7]. In particular, a partial extension of [6, Proposition 2.6] is obtained.

Section 5 contains preliminaries to Section 6. It is proved that under suitable hypotheses, the Malliavin operator D commutes with the stochastic integrals of [3]. Then, in Section 6, we prove that $u(t, x) \in \mathbb{D}^{1,2}$ - the space of Malliavin differentiable random variables with square integrable derivative. Using a standard approach, we consider a sequence of $L^2(\Omega)$ -approximations of the process $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$, $\{u_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$, $n \in \mathbb{N}$, such that they are Malliavin differentiable with uniformly bounded (in the corresponding norm) Malliavin derivatives. The section ends by establishing a SPDE satisfied by the Hilbert-valued process $Du(t, x)$ (see (6.3)). In contrast with examples where the distribution Λ is smooth (either a function or a positive measure), it seems not possible to obtain this equation by passing to the limit the sequence of SPDEs (6.5) satisfied by the Malliavin derivatives of the approximations u_n . Indeed, for this one would need to have $u(t, x) \in \mathbb{D}^{1,p}$, for some $p > 2$, a property that has not been established yet. This problem stems from the lack of $L^p(\Omega)$ estimates for the solution of (1.2) pointed out in [3]. So far, this has been only proved when σ is an affine function (see [3, Section 6]). The general case is by now an open problem.

We overcome this problem and eventually establish (6.3), by applying the operator D to Equation (1.2) and the commutation results of Section 5. The stochastic integrals in [5] and [3] are constructed assuming the property of spatial stationary covariance of the integrand. For the stochastic integral in (1.2) this follows from the “ S ” property introduced in [5] (see [3, Definition 4.4 and Lemma 4.5]). In the application of the results proved in Section 5, we need in addition the stationary covariance property for Hilbert-valued stochastic processes of the form $\{Dg(u(t, x)), (t, x) \in [0, T] \times \mathbb{R}^d\}$, where g is a smooth function. This can be achieved by considering the equation (6.1), more general than (1.2), and by proving that the

“S” property holds for the couple consisting of the solutions to these two equations.

The final Section 7 deals with the existence of density for each random variable $u(t, x)$, $(t, x) \in]0, T] \times \mathbb{R}^d$ in (1.2) when σ is constant. The results apply to the stochastic wave equation in spatial dimension $d > 4$ with an additive, Gaussian, spatially correlated noise, with covariance measure given by a Riesz kernel. This is proved by applying the Bouleau-Hirsch’s criterion. So far, we have not been able to obtain lower bounds of the dominant term of the Malliavin matrix for non-constant coefficients σ . Once more, the lack of L^p estimates we alluded before and the lack of positiveness of Λ are the reasons for such restriction.

Throughout this article, we shall use the usual convention of calling constants by the same letter, although they may vary from one expression to the other.

2 Preliminaries

In this section, we fix some notation, we present some general facts and we recall the stochastic and pathwise integrals from [3] that will be used throughout the paper. The relevant spaces are described and some relationships between them are proved.

Denote by $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support; $\mathcal{S}(\mathbb{R}^d)$ will denote the Schwartz space of rapidly decreasing \mathcal{C}^∞ functions, $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered Schwartz distributions and $\mathcal{S}'_r(\mathbb{R}^d)$ the space of tempered Schwartz distributions with rapid decrease (see [22]).

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be nonnegative, with support included in the ball of \mathbb{R}^d centered at zero and with radius one, satisfying $\int_{\mathbb{R}^d} \psi(x) dx = 1$. Set $\psi_n(x) := n^d \psi(nx)$, $n \in \mathbb{N}$. Then, as $n \rightarrow \infty$, $\psi_n \rightarrow \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{F}\psi_n \rightarrow 1$ pointwise. Moreover $|\mathcal{F}\psi_n| \leq 1$ for all $n \in \mathbb{N}$. The sequence $(\psi_n)_{n \in \mathbb{N}}$ is termed an *approximation of the identity*.

Let $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$. Denote by “*” the convolution operation. It is well-known that

$$\Lambda_n := \Lambda * \psi_n, \quad (2.1)$$

belongs to $\mathcal{S}(\mathbb{R}^d)$. Moreover,

$$|\mathcal{F}\Lambda_n(\xi)| = |\mathcal{F}\Lambda(\xi)| |\mathcal{F}\psi_n(\xi)| \leq |\mathcal{F}\Lambda(\xi)|, \quad (2.2)$$

for all $\xi \in \mathbb{R}^d$, and $\mathcal{F}\Lambda_n \rightarrow \mathcal{F}\Lambda$ pointwise as $n \rightarrow \infty$.

Let $\{F(\phi); \phi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a Gaussian process with mean zero and covariance functional

$$\mathbb{E}[F(\phi)F(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} (\phi(t) * \tilde{\psi}(t))(x) \Gamma(dx) dt, \quad (2.3)$$

where $\tilde{\psi}(t, x) := \psi(t, -x)$ and Γ is a nonnegative, nonnegative definite, tempered measure on \mathbb{R}^d . There exists a nonnegative tempered measure μ on \mathbb{R}^d such that

$\mathcal{F}\mu = \Gamma$ (see for instance [22], Chapter VII, Théorème XVIII). Then by Parseval's identity, the right-hand side of (2.3) is equal to

$$\int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\phi(t)(\xi) \overline{\mathcal{F}\psi(t)}(\xi) \mu(d\xi) dt.$$

As is explained in [4], the process F can be extended to a worthy martingale measure $M = (M_t(A); t \in \mathbb{R}_+, A \in \mathcal{B}_b(\mathbb{R}^d))$ where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the σ -field of bounded Borel sets of \mathbb{R}^d . The natural filtration generated by this martingale measure will be denoted in the sequel by $(\mathcal{F}_t)_{t \geq 0}$.

A stochastic integration theory with respect to martingale measures has been developed by Métivier and Pellaumail and by Walsh, among others. Here, we shall use [24] as reference. Using this integral, we have

$$F(\phi) = \int_0^T \int_{\mathbb{R}^d} \phi(s, z) M(ds, dz), \quad (2.4)$$

for $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$.

Extensions of the stochastic integral given in [24] have been introduced in [5] and more recently, in [3]. Throughout this article, we shall refer mainly to the latter.

Fix $T > 0$. For stochastic processes g, h satisfying suitable conditions, we define the inner products

$$\langle g, h \rangle_+ = \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (|g(s)| * |\tilde{h}(s)|)(x) \Gamma(dx) ds \right]$$

and

$$\begin{aligned} \langle g, h \rangle_0 &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (g(s) * \tilde{h}(s))(x) \Gamma(dx) ds \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \mathcal{F}g(s)(\xi) \overline{\mathcal{F}h(s)(\xi)} \mu(d\xi) ds \right]. \end{aligned}$$

The corresponding norms $\|\cdot\|_+$ and $\|\cdot\|_0$ are defined in the usual way.

Let \mathcal{P}_+ be the set of predictable processes g such that $\|g\|_+ < \infty$. In [24, Exercise 2.5] it is shown that \mathcal{P}_+ is complete and hence it is a Hilbert space. Let \mathcal{E} denote the set of *simple processes* g , that is, stochastic processes of the form

$$g(t, x; \omega) = \sum_{j=1}^m 1_{(a_j, b_j]}(t) 1_{A_j}(x) X_j(\omega), \quad (2.5)$$

for some $m \in \mathbb{N}$, where $0 \leq a_j < b_j \leq T$, $A_j \in \mathcal{B}_b(\mathbb{R}^d)$ and X_j is a bounded and \mathcal{F}_{a_j} -measurable random variable for all $1 \leq j \leq n$.

According to [24, Proposition 2.3], \mathcal{E} is dense in \mathcal{P}_+ . Hence, we can also define \mathcal{P}_+ as the completion of \mathcal{E} with respect to $\|\cdot\|_+$.

Following [5], we denote by \mathcal{P}_0 the completion of \mathcal{E} with respect to $\|\cdot\|_0$. This is a Hilbert space consisting of predictable processes which contains tempered distributions in the x -argument (whose Fourier transform are functions, \mathbb{P} -a.s.). The norm in this space is given by

$$\|g\|_0^2 = \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g(s)(\xi)|^2 \mu(d\xi) ds \right]. \quad (2.6)$$

For sufficiently smooth elements of \mathcal{P}_0 , this norm can be also written as

$$\|g\|_0^2 = \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (g(s, \cdot) * \tilde{g}(s, \cdot))(z) \Gamma(dz) ds \right]. \quad (2.7)$$

Note that \mathcal{P}_0 is not defined as the set of predictable processes for which $\|\cdot\|_0 < \infty$. In fact, it can be shown that the latter space is not complete. Since $\|\cdot\|_0 \leq \|\cdot\|_+$, we clearly have $\mathcal{P}_+ \subseteq \mathcal{P}_0$. Later, we will see that this inclusion is strict.

Consider the subsets of \mathcal{P}_+ and \mathcal{P}_0 consisting of deterministic processes, denoted by $\mathcal{P}_{+,d}$ and $\mathcal{P}_{0,d}$, respectively. In the next lemma, we give an equivalent definition of $\mathcal{P}_{0,d}$. For this, we first introduce a new space \mathcal{E}_0 consisting of Schwartz functions endowed with the inner product

$$\langle \phi, \psi \rangle_0 = \int_{\mathbb{R}^d} (\phi * \tilde{\psi})(x) \Gamma(dx) = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi) \overline{\mathcal{F}\psi(\xi)} \mu(d\xi), \quad (2.8)$$

where $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Let \mathcal{H} denote the completion of $(\mathcal{E}_0, \langle \cdot, \cdot \rangle_0)$ and set $\mathcal{H}_T := L^2([0, T]; \mathcal{H})$. In the sequel we will also denote by $\|\cdot\|_{\mathcal{H}_T}$ the norm in this space derived from the scalar product $\langle \cdot, \cdot \rangle_0$.

Lemma 2.1 *The spaces $\mathcal{P}_{0,d}$ and \mathcal{H}_T coincide.*

Proof. First, we prove the inclusion $\mathcal{H}_T \subseteq \mathcal{P}_{0,d}$. For this, let \mathcal{E}_s be the set of functions $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ which are a step function in the first argument and a Schwartz function in the second one. Notice that \mathcal{E}_s is dense in \mathcal{H}_T with respect to the norm $\|\cdot\|_{\mathcal{H}_T}$. We will show that $\mathcal{E}_s \subseteq \mathcal{P}_{+,d} \subseteq \mathcal{P}_{0,d}$, yielding the statement.

Indeed, fix $f \in \mathcal{E}_s$. Due to Leibniz' formula (see [23, Exercise 26.4]), the function $z \mapsto (|f(s, \cdot)| * |\tilde{f}(s, \cdot)|)(z)$ decreases faster than any polynomial in $|z|^{-1}$. Since Γ is a tempered measure, we have

$$\|f\|_+^2 = \int_0^T \int_{\mathbb{R}^d} (|f(s, \cdot)| * |\tilde{f}(s, \cdot)|)(z) \Gamma(dz) ds < \infty.$$

This proves the claim.

Next, we consider the set \mathcal{E}_d consisting of deterministic simple functions, and we prove that $\mathcal{E}_d \subseteq \mathcal{H}_T$. By taking closures in the norm $\|\cdot\|_{\mathcal{H}_T}$, we will obtain the inclusion $\mathcal{P}_{0,d} \subseteq \mathcal{H}_T$.

Let $g \in \mathcal{E}_d$ be given by $g = 1_{(a,b]} 1_A$, with $0 \leq a < b \leq T$. This function satisfies

$$\|g\|_{\mathcal{H}_T}^2 = \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g(s)(\xi)|^2 \mu(d\xi) ds < \infty. \quad (2.9)$$

Indeed, by writing $\|g\|_{\mathcal{H}_T}^2$ as in the right-hand side of (2.7), we have

$$\begin{aligned} \|g\|_{\mathcal{H}_T}^2 &= \int_0^T \int_{\mathbb{R}^d} ((1_{(a,b]}(s) 1_A(\cdot)) * (1_{(a,b]}(s) \widetilde{1}_A(\cdot)))(z) \Gamma(dz) ds \\ &= \int_0^T 1_{(a,b]}(s) ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_A(y) 1_A(y - z) dy \Gamma(dz) \\ &\leq (b - a)|A| \int_{\mathbb{R}^d} 1_B(z) \Gamma(dz) \leq (b - a)|A|\Gamma(\bar{B}), \end{aligned}$$

where $|A|$ denotes the Lebesgue measure of A , B stands for the ball in \mathbb{R}^d centered at 0 and with radius $\text{diam}(A) = \sup\{d(x, z); x, y \in A\}$, and \bar{B} denotes its closure in the Euclidean norm. Since Γ is a nonnegative tempered measure, it has the form $\Gamma(dz) = p(z)\nu(dz)$, where p is a polynomial and ν is a finite measure (see [22, p. 242]). Hence Γ is σ -finite. This fact along with the preceding inequalities, yields (2.9).

For an approximation of the identity $(\psi_n)_{n \in \mathbb{N}}$, we define $g_n(s) := g(s) * \psi_n \in \mathcal{S}(\mathbb{R}^d)$, $s \in [0, T]$, $n \in \mathbb{N}$. Clearly, $g_n \in \mathcal{E}_0 \subseteq \mathcal{H}_T$. Moreover, we will prove that

$$\lim_{n \rightarrow \infty} \|g_n - g\|_{\mathcal{H}_T} = 0. \quad (2.10)$$

This yields $g \in \mathcal{H}_T$.

For the proof of (2.10), we notice that by the very definition of the norm in \mathcal{H}_T ,

$$\begin{aligned} \|g_n - g\|_{\mathcal{H}_T}^2 &= \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g_n(s)(\xi) - \mathcal{F}g(s)(\xi)|^2 \mu(d\xi) ds \\ &= \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g(s)(\xi)|^2 |\mathcal{F}\psi_n(\xi) - 1|^2 \mu(d\xi) ds. \end{aligned} \quad (2.11)$$

By using (2.9) and applying bounded convergence, the last term converges to zero as $n \rightarrow \infty$. \square

In order to introduce notation and provide some introductory material, we give a brief overview of the integrals defined in [3]. In the next section, we shall extend these integrals to Hilbert-valued stochastic processes.

Let $Z = \{Z(t, x); t \in [0, T], x \in \mathbb{R}^d\}$ be a real-valued random field with the following properties.

(A1) Z is a predictable process satisfying $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[Z(t, x)^2] < \infty$.

(A2) Z has *spatial stationary covariance*. That is, for any $t \in [0, T]$, $x, y \in \mathbb{R}^d$,

$$\mathbb{E}[Z(t, x)Z(t, x + y)] = \mathbb{E}[Z(t, 0)Z(t, y)] =: \gamma_t^Z(y).$$

The process

$$M_t^Z(A) := \int_0^t \int_A Z(s, z) M(ds, dz), \quad t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d),$$

defines a worthy martingale measure (see [24]).

Similarly to the definition of the norms $\|\cdot\|_+$ and $\|\cdot\|_0$, for deterministic functions g sufficiently smooth, we set

$$\begin{aligned} \|g\|_{+,Z}^2 &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (|g(s, \cdot) Z(s, \cdot)| * |\tilde{g}(s, \cdot) \tilde{Z}(s, \cdot)|)(z) \Gamma(dz) ds \right], \\ \|g\|_{0,Z}^2 &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} ((g(s, \cdot) Z(s, \cdot)) * (\tilde{g}(s, \cdot) \tilde{Z}(s, \cdot)))(z) \Gamma(dz) ds \right]. \end{aligned} \quad (2.12)$$

Let $\mathcal{P}_{+,Z}$ and $\mathcal{P}_{0,Z}$ denote the completion of $(\mathcal{E}, \|\cdot\|_{+,Z})$ and $(\mathcal{E}, \|\cdot\|_{0,Z})$ with respect to these norms, respectively. Accordingly to [24, Exercise 2.5] $\mathcal{P}_{+,Z}$ is exactly the set of all predictable processes ϕ for which $\|\phi\|_{+,Z} < \infty$. However, for $\mathcal{P}_{0,Z}$ there is no similar characterization.

By Bochner's Theorem ([22, Chapter VII, Théorème XVIII]), there exists a non-negative tempered measure ν_t^Z such that $\gamma_t^Z = \mathcal{F}\nu_t^Z$, where γ_t^Z is defined in **(A2)**. Moreover, we have

$$\gamma_t^Z \Gamma = (\mathcal{F}\nu_t^Z)(\mathcal{F}\mu) = \mathcal{F}(\mu * \nu_t^Z). \quad (2.13)$$

In the sequel we set $\mu_t^Z := \mu * \nu_t^Z$. Due to Fubini's Theorem, assumption **(A2)**, (2.13) and Parseval's Identity, we see that for all $g \in \mathcal{P}_{0,d}$

$$\begin{aligned} \|g\|_{0,Z}^2 &= \int_0^T \int_{\mathbb{R}^d} (g(s, \cdot) * \tilde{g}(s, \cdot))(z) \gamma_s^Z(z) \Gamma(dz) ds \\ &= \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g(s)(\xi)|^2 \mu_s^Z(d\xi) ds. \end{aligned} \quad (2.14)$$

Following [3], we describe the assumptions on deterministic functions that may be integrated with respect to the martingale measure M^Z . We shall denote by ψ an arbitrary function in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ satisfying the assumptions described at the beginning of this section.

(A3) $t \mapsto \Lambda(t)$ is a deterministic function with values in $\mathcal{S}'_r(\mathbb{R}^d)$; the mapping $(t, \xi) \mapsto \mathcal{F}\Lambda(t)(\xi)$ is measurable and

$$\int_0^T \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds < \infty.$$

(A4) For all ψ and all $0 \leq a \leq b \leq T$, we have

$$\int_a^b (\Lambda(s) * \psi)(x) ds \in \mathcal{S}(\mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d} \int_a^b |(\Lambda(s) * \psi)(x)| ds dx < \infty.$$

(A5) $\mathcal{F}\Lambda(t)$ is such that

$$\lim_{h \downarrow 0} \int_0^T \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{s < r < s+h} |\mathcal{F}\Lambda(r)(\xi + \eta) - \mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds = 0.$$

Notice that **(A3)** implies

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi)|^2 \mu(d\xi) ds < \infty. \quad (2.15)$$

In [3, Theorem 3.1] it is proved that under assumptions **(A1)**, **(A2)**, **(A3)**, and either **(A4)** or **(A5)**, $\Lambda \in \mathcal{P}_{0,Z}$ and that the stochastic integral $((\Lambda \cdot M^Z)_t; t \in [0, T])$ is well-defined as a real-valued square-integrable martingale. Moreover,

$$\mathbb{E}[(\Lambda \cdot M^Z)_t^2] = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi)|^2 \mu_s^Z(d\xi) ds = \|\Lambda\|_{0,Z}^2. \quad (2.16)$$

Using Assumption **(A2)** and the identities $\nu_s^Z(\mathbb{R}^d) = \gamma_s^Z(0) = \mathbb{E}[Z(s, 0)^2]$, one can obtain the following upper bound:

$$\begin{aligned} \mathbb{E}[(\Lambda \cdot M^Z)_t^2] &= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi)|^2 \mu_s^Z(d\xi) ds \\ &\leq \int_0^t \nu_s^Z(\mathbb{R}^d) \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds \\ &= \int_0^t \mathbb{E}[Z(s, 0)^2] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds, \end{aligned} \quad (2.17)$$

Owing to **(A1)**, this yields

$$\mathbb{E}[(\Lambda \cdot M^Z)_t^2] \leq C \int_0^t \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds. \quad (2.18)$$

Let $g \in L^2([0, T]; L^1(\mathbb{R}^d))$. For a stochastic process Z satisfying the conditions **(A1)**, **(A2)**, and following again [3], we introduce the norm

$$\|g\|_{1,Z}^2 := \mathbb{E} \left[\int_0^T \left(\int_{\mathbb{R}^d} g(s, z) Z(s, z) dz \right)^2 ds \right].$$

Proceeding as in the derivation of (2.14), we obtain

$$\begin{aligned}
\|g\|_{1,Z}^2 &:= \mathbb{E} \left[\int_0^T \left(\int_{\mathbb{R}^d} g(s, z) Z(s, z) dz \right) \left(\int_{\mathbb{R}^d} g(s, y) Z(s, y) dy \right) ds \right] \\
&= \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) g(s, y) \gamma_s^Z(z - y) dy dz ds \\
&= \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g(s)(\eta)|^2 \nu_s^Z(d\eta) ds.
\end{aligned} \tag{2.19}$$

The closure of the space \mathcal{E} with respect to the norm $\|\cdot\|_{1,Z}$ is denoted by $\mathcal{P}_{1,Z}$.

In order to give a rigorous meaning to pathwise convolutions, some additional assumptions are needed. The are described as follows.

(A6) The mapping $t \mapsto \Lambda(t)$ is a deterministic function with values in $\mathcal{S}'_r(\mathbb{R}^d)$ and satisfies

$$\int_0^T \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds < \infty.$$

(A7) The mapping $t \mapsto \mathcal{F}\Lambda(t)$ is a deterministic function with values in $\mathcal{S}'_r(\mathbb{R}^d)$ and such that

$$\lim_{h \downarrow 0} \int_0^T \sup_{\eta \in \mathbb{R}^d} \sup_{s < r < s+h} |\mathcal{F}\Lambda(r)(\eta) - \mathcal{F}\Lambda(s)(\eta)|^2 ds = 0.$$

Note that these two conditions coincide respectively with **(A3)** and **(A5)** if $\mu = \delta_0$.

Assume **(A1)**, **(A2)**, **(A6)** and either **(A4)** or **(A7)**. In [3, Proposition 3.4] it is proved that for any $x \in \mathbb{R}^d$,

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(s, z) Z(s, z) dz ds, \quad t \in [0, T], \tag{2.20}$$

defines a stochastic process with values in $L^2(\Omega)$. In addition, from (2.19) and **(A1)**, **(A2)**, it follows that

$$\begin{aligned}
\|\Lambda\|_{1,Z}^2 &= \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(s, x) Z(s, x) dx ds \right)^2 \right] \\
&\leq \int_0^t \nu_s^Z(\mathbb{R}^d) \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds \\
&\leq C \int_0^t \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds.
\end{aligned} \tag{2.21}$$

In this article, we will refer extensively to the Hilbert space $L^2(\Omega; \mathcal{H}_T) = L^2([0, T] \times \Omega; \mathcal{H})$. In [6, Proposition 2.6], it is proved that $\mathcal{P}_0 \subseteq L^2(\Omega; \mathcal{H}_T)$. Then if $g \in \mathcal{P}_0$, $\|g\|_0 = \|g\|_{L^2(\Omega; \mathcal{H}_T)}$ with $\|\cdot\|_0$ defined as in (2.6).

Using Lemma 2.1, the expression (2.4) holds for any $\phi \in \mathcal{H}_T$. As a consequence, we see that $F = (F(\phi); \phi \in \mathcal{H}_T)$ is an isonormal Gaussian process. In this article, we will use the theory of Malliavin calculus based on the Gaussian process F (see [14]).

The process F can be identified with an \mathcal{H} -valued cylindrical Wiener process, as follows. As it is shown in [4], by an approximation procedure we define $W_t(\psi) = F(1_{[0,t]} \psi)$, $t \in [0, T]$, $\psi \in \mathcal{H}$. Consider a complete orthonormal system (CONS) of \mathcal{H} that we denote by $(g_k)_{k \in \mathbb{N}}$. Then,

$$W = \{W^k(t) := W_t(g_k), t \in [0, T], k \in \mathbb{N}\}$$

defines a sequence of independent standard Brownian motions.

Conversely, the process $(F(\phi) = \sum_{k \in \mathbb{N}} \int_0^T \langle \phi, e_k \rangle_{\mathcal{H}_T} dW^k(t), \phi \in \mathcal{H}_T)$ is an isonormal Gaussian process.

Let $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mu})$ be the canonical space of a standard real-valued Brownian motion on $[0, T]$. In the sequel, we will use the reference probability space $(\Omega, \mathcal{G}, \mathbb{P}) = (\bar{\Omega}^{\mathbb{N}}, \bar{\mathcal{G}}^{\otimes \mathbb{N}}, \bar{\mu}^{\otimes \mathbb{N}})$. Consider the Hilbert space \mathbb{H} consisting of sequences $(h^k)_{k \in \mathbb{N}}$ of functions $h_k : [0, T] \rightarrow \mathbb{R}$ which are absolutely continuous with respect to the Lebesgue measure and such that $\sum_{k \in \mathbb{N}} \int_0^T |\dot{h}^k(s)|^2 ds < \infty$, where \dot{h}^k refers to the derivative of h^k defined almost everywhere. Then, $(\Omega, \mathbb{H}, \mathbb{P})$ is an abstract Wiener space.

Frequently, we will use the notation

$$J_1(s) := \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) \quad (2.22)$$

and

$$J_2(s) := \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2, \quad (2.23)$$

$s \in [0, T]$.

If the assumptions **(A3)**, **(A6)**, respectively, are satisfied, then

$$\int_0^T J_1(s) ds < \infty, \quad \int_0^T J_2(s) ds < \infty, \quad (2.24)$$

respectively.

3 Hilbert-valued stochastic integrals

In this section, we develop an extension of the integrals introduced in [3] to Hilbert-valued integrands. For similar results in the setting of [5], we refer the reader to [18].

Let \mathcal{A} be a separable real Hilbert space with inner-product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{A}}$ respectively. In the sequel, $(\varphi_k)_{k \in \mathbb{N}}$ will denote a complete orthonormal

system of \mathcal{A} . Let $\{Z(t, x), t \in [0, T], x \in \mathbb{R}^d\}$ be an \mathcal{A} -valued random field satisfying the following properties similar to **(A1)**, **(A2)**:

(A8) Z is an \mathcal{A} -valued predictable process satisfying

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[\|Z(t, x)\|_{\mathcal{A}}^2] < \infty.$$

(A9) Z has a spatial stationary covariance function coordinatewise. That is, for all $k \in \mathbb{N}$ and $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\mathbb{E}[Z_k(t, x)Z_k(t, x + y)] = \mathbb{E}[Z_k(t, 0)Z_k(t, y)] =: \gamma_{k,t}^Z(y),$$

where $Z_k(t, x) := \langle Z(t, x), \varphi_k \rangle_{\mathcal{A}}$.

For each $k \in \mathbb{N}$, the stochastic process

$$M_t^{Z_k}(A) := \int_0^t \int_A Z_k(s, z) M(ds, dz), \quad t \in [0, T], \quad A \in \mathcal{B}_b(\mathbb{R}^d),$$

defines a real-valued worthy martingale measure.

According to [18],

$$M_t^Z(A) := \sum_{k \in \mathbb{N}} M_t^{Z_k}(A) \varphi_k, \quad t \in [0, T], \quad A \in \mathcal{B}_b(\mathbb{R}^d),$$

defines a \mathcal{A} -valued worthy martingale measure and by construction, $\langle M_t^Z(A), \varphi_k \rangle_{\mathcal{A}} = M_t^{Z_k}(A)$. Moreover,

$$\mathbb{E}[\|M_t^Z(A)\|_{\mathcal{A}}^2] \leq C_{t,A} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[\|Z(t, x)\|_{\mathcal{A}}^2],$$

(see [18, p. 5]).

For an \mathcal{A} -valued stochastic process g , we set $g_k = \langle g, \varphi_k \rangle_{\mathcal{A}}$, and define

$$\|g\|_{0,\mathcal{A}}^2 := \sum_{k \in \mathbb{N}} \|g_k\|_0^2.$$

Also, for a deterministic function h , we define

$$\|h\|_{0,Z,\mathcal{A}}^2 := \sum_{k \in \mathbb{N}} \|h\|_{0,Z_k}^2, \quad \|h\|_{+,Z,\mathcal{A}}^2 := \sum_{k \in \mathbb{N}} \|h\|_{+,Z_k}^2, \quad \|h\|_{1,Z,\mathcal{A}}^2 := \sum_{k \in \mathbb{N}} \|h\|_{1,Z_k}^2.$$

Let $\mathcal{E}_{\mathcal{A}}$ be the set of \mathcal{A} -valued simple processes. That is, processes with a similar expression as in (2.5), where X_j , $j = 1, \dots, m$, are \mathcal{A} -valued random variables. Then, we denote by $\mathcal{P}_{0,Z,\mathcal{A}}$, $\mathcal{P}_{+,Z,\mathcal{A}}$ and $\mathcal{P}_{1,Z,\mathcal{A}}$ the completions of \mathcal{E} with respect to $\|\cdot\|_{0,Z,\mathcal{A}}$, $\|\cdot\|_{+,Z,\mathcal{A}}$ and $\|\cdot\|_{1,Z,\mathcal{A}}$, respectively.

Theorem 3.1 Let $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ be an \mathcal{A} -valued stochastic process satisfying conditions **(A8)**, **(A9)**. Let $t \mapsto \Lambda(t)$ be a deterministic function taking values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. We suppose that **(A3)** and either **(A4)** or **(A5)** are satisfied. Then $\Lambda \in \mathcal{P}_{0, Z, \mathcal{A}}$ and the stochastic integral $\{(\Lambda \cdot M^Z)_t, t \in [0, T]\}$ is well-defined as an \mathcal{A} -valued square integrable process. Moreover,

$$\begin{aligned} \mathbb{E}[\|(\Lambda \cdot M^Z)_t\|_{\mathcal{A}}^2] &= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi)|^2 \mu_s^Z(d\xi) ds = \|\Lambda\|_{0, Z, \mathcal{A}}^2 \\ &\leq \int_0^t \mathbb{E}[\|Z(s, 0)\|_{\mathcal{A}}^2] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds \\ &\leq C \int_0^t \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds \end{aligned} \quad (3.1)$$

Proof. From [3, Theorem 3.1] we know that $\Lambda \in \mathcal{P}_{0, Z_k}$ and also that $\{(\Lambda \cdot M^{Z_k})_t, t \in [0, T]\}$ is well-defined for any $k \in \mathbb{N}$. In addition,

$$\mathbb{E}[\|(\Lambda \cdot M^{Z_k})_t\|^2] = \|\Lambda\|_{0, Z_k}^2.$$

The proof of $\Lambda \in \mathcal{P}_{0, Z, \mathcal{A}}$ follows the same arguments as in [3, Theorem 3.1]. Firstly, we check that Λ_n (defined in (2.1)) belongs to $\mathcal{P}_{0, Z, \mathcal{A}}$, and then that

$$\lim_{n \rightarrow \infty} \|\Lambda_n - \Lambda\|_{0, Z, \mathcal{A}}^2 = 0.$$

The arguments of [3, Theorem 3.1] can be adapted by using **(A3)**, **(A8)** and the following remark: For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\|\varphi\|_{0, Z_k}^2 \leq \int_0^T E(|Z^k(s, 0)|^2) \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\varphi(s)(\xi + \eta)|^2.$$

It follows that

$$\begin{aligned} \|\varphi\|_{0, Z, \mathcal{A}}^2 &= \sum_{k \in \mathbb{N}} \|\varphi\|_{0, Z_k}^2 \\ &\leq \int_0^T E(\|Z(s, 0)\|_{\mathcal{A}}^2) \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\varphi(s)(\xi + \eta)|^2. \end{aligned}$$

For any $t \in [0, T]$, define

$$(\Lambda \cdot M^Z)_t = \sum_{k \in \mathbb{N}} (\Lambda \cdot M^{Z_k})_t \varphi_k.$$

Clearly,

$$\mathbb{E}[\|(\Lambda \cdot M^Z)_t\|_{\mathcal{A}}^2] = \sum_{k \in \mathbb{N}} \mathbb{E}[\|(\Lambda \cdot M^{Z_k})_t\|^2] = \sum_{k \in \mathbb{N}} \|\Lambda\|_{0, Z_k}^2 = \|\Lambda\|_{0, Z, \mathcal{A}}^2.$$

The estimates (3.1) follows from (2.17) applied to each stochastic integral $\Lambda \cdot M^{Z_k}$, $k \in \mathbb{N}$, along with **(A8)**. \square

By using similar arguments as in the proof of [3, Proposition 3.4], one can also give an extension of the pathwise integral to \mathcal{A} -valued stochastic processes. For this, it is worthy noticing that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} \|\varphi\|_{1,Z,\mathcal{A}}^2 &= \sum_{k \in \mathbb{N}} \|\varphi\|_{1,Z^k}^2 \\ &\leq \int_0^t \mathbb{E}(\|Z(s, 0)\|_{\mathcal{A}}^2) \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\varphi(s)(\eta)|^2 ds. \end{aligned}$$

The extension reads as follows.

Theorem 3.2 *Let $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ be a stochastic process as in Theorem 3.1. Let $t \mapsto \Lambda(t)$ be a deterministic function taking values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. We suppose that **(A6)** and either **(A4)** or **(A7)** are satisfied. Then*

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(s, z) Z(s, z) dz ds := \sum_{k \in \mathbb{N}} \left(\int_0^t \int_{\mathbb{R}^d} \Lambda(s, z) Z_k(s, z) dz ds \right) \varphi_k,$$

$t \in [0, T]$, defines a stochastic process with values in $L^2(\Omega; \mathcal{A})$. Moreover,

$$\begin{aligned} \mathbb{E} \left[\left\| \int_0^t \int_{\mathbb{R}^d} \Lambda(s, z) Z(s, z) dz ds \right\|_{\mathcal{A}}^2 \right] &= \sum_{k \in \mathbb{N}} \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(s, z) Z_k(s, z) dz ds \right)^2 \right] \\ &= \|\Lambda\|_{1,Z,\mathcal{A}}^2 \\ &\leq \sum_{k \in \mathbb{N}} \int_0^t \mathbb{E}[Z_k(s, 0)^2] \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds \\ &\leq C \int_0^t \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds. \end{aligned} \tag{3.2}$$

4 Equivalence of stochastic integrals

In this section, we consider a particular case of integrands described as follows. Let Z be a stochastic process satisfying **(A1)**, **(A2)**. Let $t \in [0, T] \mapsto \Lambda(t) \in \mathcal{S}'_r(\mathbb{R}^d)$. We are interested in stochastic processes which are obtained as the limit in the topology of \mathcal{P}_0 of

$$\Phi_{t,x}^n := \Lambda_n(t - \cdot, x - *) Z(\cdot, *), n \in \mathbb{N},$$

where $(t, x) \in [0, T] \times \mathbb{R}^d$ is fixed and Λ_n is given in (2.1).

For this class of integrands, later denoted by $\Phi_{t,x}$ (and also at some places by $\Lambda(t - \cdot, x - *)Z(\cdot, *)$, by an abuse of language), we prove that the integrals in the Conus-Dalang sense ([3]) and with respect to the \mathcal{H} -valued cylindrical Wiener process $(W_t, t \in [0, T])$ (see for instance [7]), coincide with the divergence operator (also termed Skorohod integral) of Malliavin Calculus (see [14, Section 1.3]). For this, we need further insight on the relationships between the spaces \mathcal{P}_0 , $\mathcal{P}_{0,Z}$ and $L^2(\Omega, \mathcal{H}_T)$ introduced in Section 2.

We notice that, for Schwartz functions $g \in \mathcal{S}(\mathbb{R}^d)$ and a process Z satisfying **(A1)**, **(A2)**,

$$\|gZ\|_0 = \|g\|_{0,Z} = \|gZ\|_{L^2(\Omega; \mathcal{H}_T)}. \quad (4.1)$$

To simplify the notation, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, we write $[\Lambda_n Z]^{t,x}$ to denote the stochastic process $(\Lambda_n(t - \cdot, x - *)Z(\cdot, *))$, where “.” and “*” denote the time and space arguments, respectively.

Lemma 4.1 *Let Z be a stochastic process satisfying the hypotheses **(A1)**, **(A2)**. Let $t \in [0, T] \mapsto \Lambda(t)$ be a function satisfying the assumptions **(A3)** and either **(A4)** or **(A5)**. Fix $(t, x) \in [0, T] \times \mathbb{R}^d$. Then, for all $n \in \mathbb{N}$,*

1. $\Lambda_n, \Lambda \in \mathcal{P}_{0,Z}$,
2. $[\Lambda_n Z]^{t,x} \in \mathcal{P}_+$,
3. The sequence $([\Lambda_n Z]^{t,x})_{n \in \mathbb{N}}$ converges in \mathcal{P}_0 to an element $\Phi_{t,x} \in \mathcal{P}_0$, and

$$\|\Phi_{t,x}\|_0 = \|\Lambda(t - \cdot, x - *)\|_{0,Z}. \quad (4.2)$$

Moreover, $\Phi_{t,x} \in L^2(\Omega; \mathcal{H}_T)$ and

$$\|\Phi_{t,x}\|_0 = \|\Lambda(t - \cdot, x - *)\|_{L^2(\Omega; \mathcal{H}_T)}.$$

Proof. The assertions of part (1) are shown in [3, Theorem 3.1]. By the definition of the norm $\|\cdot\|_+$, we have

$$\begin{aligned} & \|\Lambda_n(t - \cdot, x - *)Z(\cdot, *)\|_+^2 \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\Lambda_n(t - s, x - z)| |\Lambda_n(t - s, x - y + z)| \right. \\ & \quad \times |Z(s, y)| |Z(s, y - z)| dy \Gamma(dz) ds \left. \right] \\ & \leq \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[Z(r, y)^2] \\ & \quad \times \int_0^t \int_{\mathbb{R}^d} (|\Lambda_n(t - s, x - \cdot)| * |\tilde{\Lambda}_n(t - s, x - \cdot)|)(z) \Gamma(dz) ds. \end{aligned} \quad (4.3)$$

Since $\Lambda_n \in \mathcal{S}(\mathbb{R}^d)$ and Γ is a tempered measure, by using Leibniz' rule ([23, Exercise 26.4]) the last expression is finite. This proves (ii).

For $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
& \|\Lambda_n(t - \cdot, x - *) - \Lambda_m(t - \cdot, x - *)\|_{0,Z}^2 \\
&= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi) - \mathcal{F}\Lambda_m(t-s)(\xi)|^2 \mu_s^Z(d\xi) ds \\
&= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\xi)|^2 |\mathcal{F}\psi_n(\xi) - \mathcal{F}\psi_m(\xi)|^2 \mu_s^Z(d\xi) ds. \tag{4.4}
\end{aligned}$$

By bounded convergence, this converges to zero as $n, m \rightarrow \infty$. Using (4.1), we conclude that $([\Lambda_n Z]^{t,x})_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{P}_0 . Let us denote by $\Phi_{t,x}$ its limit. Using similar computations as in (4.4) with Λ_m replaced by Λ , and since $\Lambda \in \mathcal{P}_{0,Z}$, we have,

$$\lim_{n \rightarrow \infty} \|\Lambda_n(t - \cdot, x - *)\|_{0,Z} = \|\Lambda(t - \cdot, x - *)\|_{0,Z}.$$

Thus,

$$\|\Phi_{t,x}(\cdot, *)\|_0 = \lim_{n \rightarrow \infty} \|\Lambda_n(t - \cdot, x - *)\|_{0,Z} = \|\Lambda(t - \cdot, x - *)\|_{0,Z}.$$

Since $\mathcal{P}_0 \subseteq L^2(\Omega; \mathcal{H}_T)$ (see [6, Proposition 2.6]), we conclude that $\Phi_{t,x} \in L^2(\Omega; \mathcal{H}_T)$. \square

The preceding lemma admits easily an extension to Hilbert-valued stochastic processes. Next, we consider a particular example of such an extension for processes that are Malliavin derivatives.

Lemma 4.2 *The function Λ and the stochastic process Z are as in Lemma 4.1. Moreover, we assume that the \mathcal{H}_T -valued stochastic process DZ satisfies (A8), (A9). Fix $(t, x) \in [0, T] \times \mathbb{R}^d$, Then, by setting*

$$[\Lambda_n DZ]^{t,x} := (\Lambda_n(t-s, x-y) DZ(s, y), (s, y) \in [0, T] \times \mathbb{R}^d),$$

the following holds.

1. $\Lambda_n, \Lambda \in \mathcal{P}_{0,DZ(t,x), \mathcal{H}_T}$,
2. $[\Lambda_n DZ]^{t,x} \in \mathcal{P}_{+, \mathcal{H}_T}$
3. The sequence $([\Lambda_n DZ]^{t,x})_{n \in \mathbb{N}}$ converges in $\mathcal{P}_{0, \mathcal{H}_T}$ to a \mathcal{H}_T -valued stochastic process $\Phi_{t,x}^{(1)} := \{\Phi_{t,x}^{(1)}(s, y), (s, y) \in [0, T] \times \mathbb{R}^d\}$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\|\Phi_{t,x}^{(1)}\|_{0, \mathcal{H}_T} = \|\Lambda(t - \cdot, x - *)\|_{0, DZ, \mathcal{H}_T}. \tag{4.5}$$

4. $\Phi_{t,x}^{(1)} = D\Phi_{t,x}$, where $\Phi_{t,x}$ is the process defined in part 3 of Lemma 4.1.

Proof. Statement 1 can be shown as in [3, Theorem 3.1] with the tools provided in Section 3. For the proof of part 2, we follow similar computations as in (4.3) to obtain

$$\begin{aligned} \|\Lambda_n(t - \cdot, x - *)DZ_{\cdot,*}(t, x)\|_{+, \mathcal{H}_T}^2 &\leq C \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[\|DZ(r, y)\|_{\mathcal{H}_T}^2] \\ &\times \int_0^t \int_{\mathbb{R}^d} (|\Lambda_n(t-s, x - \cdot)| * |\tilde{\Lambda}_n(t-s, x - \cdot)|)(z) \Gamma(dz) ds. \end{aligned}$$

By the same arguments as in Lemma 4.1, this last expression is finite.

Similarly as for the statement 3 of Lemma 4.1, we prove that $([\Lambda_n DZ]^{t,x}, n \in \mathbb{N})$ is a Cauchy sequence in the norm $\|\cdot\|_{0,DZ, \mathcal{H}_T}$ and that its limit $\Phi^{(1)}$ satisfies (4.5).

As for part 4, we notice that by Lemma 4.1, the sequence $(\Lambda_n(t - \cdot, x - *)Z(\cdot, *), n \in \mathbb{N})$ converges in $L^2(\Omega; \mathcal{H}_T)$ to a random vector $\Phi_{t,x}$. Moreover, $D(\Lambda_n(t - \cdot, x - *)Z) = \Lambda_n(t - \cdot, x - *)DZ$. Hence by part 3, the sequence $(D(\Lambda_n(t - \cdot, x - *)Z))_{n \in \mathbb{N}}$ converges in $L^2(\Omega; \mathcal{H}_T^{\otimes 2})$. Since D is a closed operator, we conclude using again part 3. \square

Let Λ and Z be as in Lemma 4.1 and $(e_k)_{k \in \mathbb{N}}$ be a CONS of \mathcal{H}_T . For any $k \in \mathbb{N}$, the real-valued stochastic process $D^{e_k}Z := \langle DZ, e_k \rangle_{\mathcal{H}_T}$ satisfies the hypotheses of Lemma 4.1. Hence, the sequence $([\Lambda_n D^{e_k}Z]^{t,x})_{n \in \mathbb{N}}$ converges in \mathcal{P}_0 to an element denoted by $\Phi_{t,x}^{(e_k)}$. Since the Malliavin derivative is a closed operator, we have $D^{e_k}\Phi_{t,x} = \Phi_{t,x}^{(e_k)}$, with $\Phi_{t,x}$ given in Lemma 4.1, and $\Phi_{t,x}^{(e_k)} = \langle \Phi_{t,x}^{(1)}, e_k \rangle_{\mathcal{H}_T}$, with $\Phi_{t,x}^{(1)}$ defined in Lemma 4.2.

Let g denote a predictable stochastic process belonging to $L^2(\Omega; \mathcal{H})$. Using the stochastic integration theory developed for instance in [7], the integral of g with respect to the cylindrical Brownian motion $\{W_t, t \in [0, T]\}$ described in section 2 is well-defined, as follows:

$$(g \cdot W)_t := \int_0^t g(s) dW_s := \sum_{k \in \mathbb{N}} \int_0^t \langle g(s, *), g_k(*) \rangle_{\mathcal{H}} dW_s^k, \quad t \in [0, T], \quad (4.6)$$

where $(g_k)_{k \in \mathbb{N}}$ is a CONS of \mathcal{H} .

The next proposition provides an extension of [6, Proposition 2.6] to the stochastic integral in [3]. This is only for the class of integrands $\Phi_{t,x}$ defined in Lemma 4.1 though.

Proposition 4.3 *Let Λ fulfill **(A3)** and either **(A4)** or **(A5)**. Let Z be a stochastic process satisfying conditions **(A1)** and **(A2)**. Fix $(t, x) \in [0, T] \times \mathbb{R}^d$ and consider the stochastic process $\Phi_{t,x}$ defined in Lemma 4.1. Then*

$$(\Lambda(t - \cdot, x - *) \cdot M^Z)_t = (\Phi_{t,x} \cdot W)_t, \quad t \in [0, T].$$

where the expression on the left-hand side refers to the integral of Conus and Dalang (see [3, Theorem 3.1]), while on the right-hand side, it refers to the integral defined in (4.6).

Proof. From Lemma 4.1, part 2, we have that $[\Lambda_n Z]^{t,x} \in \mathcal{P}_+ \subseteq \mathcal{P}_0 \subseteq L^2(\Omega; \mathcal{H}_T)$. Consequently, the stochastic integral in (4.6) exists for $g := [\Lambda_n Z]^{t,x}$ and it satisfies the isometry property

$$\begin{aligned}\mathbb{E}\left[\left([\Lambda_n Z]^{t,x} \cdot W\right)_t^2\right] &= \mathbb{E}\left[\int_0^t \|\Lambda_n(t-\cdot, x-\cdot)Z(\cdot, \cdot)\|_{\mathcal{H}}^2 ds\right] \\ &= \|[\Lambda_n Z]^{t,x}\|_0^2 = \|\Lambda_n(t-\cdot, x-\cdot)\|_{0,Z}^2.\end{aligned}$$

Moreover, $(\Lambda_n(t-\cdot, x-\cdot) \cdot M^Z)_t$ is well-defined as a Walsh's stochastic integral with respect to the martingale measure M^Z . According to [6, Proposition 2.6(a)]

$$(\Lambda_n(t-\cdot, x-\cdot) \cdot M^Z)_t = ([\Lambda_n Z]^{t,x} \cdot W)_t.$$

We can now pass to the limit as $n \rightarrow \infty$ and notice that $[\Lambda_n Z]^{t,x}$ converges in $L^2(\Omega; \mathcal{H}_T)$ to $\Phi_{t,x}$. We obtain,

$$(\Phi_{t,x} \cdot W)_t := L^2(\Omega) - \lim_{n \rightarrow \infty} ([\Lambda_n Z]^{t,x} \cdot W)_t.$$

On the other hand, for the stochastic integral in [3] we have

$$(\Lambda(t-\cdot, x-\cdot) \cdot M^Z)_t = L^2(\Omega) - \lim_{n \rightarrow \infty} (\Lambda_n(t-\cdot, x-\cdot) \cdot M^Z)_t.$$

This ends the proof. \square

In the next proposition, we prove the equality between the divergence operator (also called Skorohod integral) applied to the process $\Phi_{t,x}$ and the stochastic integral $(\Phi_{t,x} \cdot W)_t$.

Proposition 4.4 *The assumptions are the same as in Proposition 4.3. Fix $(t, x) \in [0, T] \times \mathbb{R}^d$. The stochastic process $\Phi_{t,x}$ derived in Lemma 4.1 part 3 satisfies*

$$\delta(\Phi_{t,x}) = (\Phi_{t,x} \cdot W)_t, \quad (4.7)$$

where δ denotes the Skorohod integral.

Proof. We follow a similar approach as in [14, Section 1.3.2]. Let $g = 1_{(a,b]} 1_A X$, where $0 \leq a < b \leq t$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and X is a bounded and \mathcal{F}_a -measurable random variable. Assume first that $X \in \mathbb{D}^{1,2}$. Then [14, (1.44)] yields

$$\delta(g) = X F(1_{(a,b]} 1_A).$$

Since $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$ and δ is closed, this equality extends to $X \in L^2(\Omega)$.

On the other hand $(g \cdot W)_t = X F(1_{(a,b]} 1_A)$ as it is shown for instance in [6, p. 11]. By linearity of the integral operators we see that (4.7) holds for a suitable class of elementary processes.

We know that $[\Lambda_n Z]^{t,x} \in \mathcal{P}_+$. Therefore, there exists a sequence of elementary processes $(g_{n,m}^{t,x})_{m \in \mathbb{N}}$ converging to $[\Lambda_n Z]^{t,x}$ in $L^2(\Omega, \mathcal{H}_T)$ as $m \rightarrow \infty$. Since the operator δ is closed, we obtain

$$([\Lambda_n Z]^{t,x} \cdot W)_t = L^2(\Omega) - \lim_{m \rightarrow \infty} (g_{n,m}^{t,x} \cdot W)_t = L^2(\Omega) - \lim_{m \rightarrow \infty} \delta(g_{n,m}^{t,x}) = \delta([\Lambda_n Z]^{t,x}).$$

Finally, using once again that δ is closed, we have

$$(\Phi_{t,x} \cdot W)_t = L^2(\Omega) - \lim_{n \rightarrow \infty} ([\Lambda_n Z]^{t,x} \cdot W)_t = L^2(\Omega) - \lim_{n \rightarrow \infty} \delta([\Lambda_n Z]^{t,x}) = \delta(\Phi_{t,x}).$$

This finishes the proof of the proposition. \square

5 Malliavin derivatives of stochastic and pathwise integrals

In this section, we state conditions for commuting the Malliavin derivative operator with two types of integrals: the class of stochastic integrals studied in Section 4 and the pathwise integrals of [3] (see (2.20)). For the former we rely on [14, Proposition 1.3.2] and we check that the assumptions of this proposition are satisfied by the relevant integrands. As for the latter, we give a direct proof.

Throughout the section, we fix Λ satisfying the assumption **(A3)** and either **(A4)** or **(A5)** and a stochastic process Z satisfying **(A1)** and **(A2)**. For any $(t, x) \in [0, T] \times \mathbb{R}^d$, we shall consider the stochastic process $\Phi_{t,x}$ given in Lemma 4.1. We will use the notation

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) Z(s, z) M(ds, dz)$$

to refer to each of the stochastic integrals $\delta(\Phi_{t,x})$, $(\Phi_{t,x} \cdot W)_t$, $(\Lambda(t-\cdot, x-\cdot) \cdot M^Z)_t$ considered in Section 4. In fact, owing to Propositions 4.3, 4.4 they coincide.

Proposition 5.1 *We assume that Λ satisfies the assumption **(A3)** and either **(A4)** or **(A5)**. Consider a stochastic process Z satisfying **(A1)** and **(A2)** and such that for any $(t, x) \in [0, T] \times \mathbb{R}^d$, $Z(t, x) \in \mathbb{D}^{1,2}$. Suppose also that DZ fulfills the assumptions **(A8)** and **(A9)** with $\mathcal{A} = \mathcal{H}_T$. Then, for every $(t, x) \in [0, T] \times \mathbb{R}^d$,*

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) Z(s, z) M(ds, dz) \in \mathbb{D}^{1,2}$$

and

$$D \left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) Z(s, z) M(ds, dz) \right)$$

$$= \Lambda(t - \cdot, x - *)Z(\cdot, *) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)DZ(s, z)M(ds, dz), \quad (5.1)$$

where the integral in the right-hand side of (5.1) is the Hilbert-valued stochastic integral $\Lambda(t - \cdot, x - *) \cdot M^{DZ}$ given in Theorem 3.1.

Proof. From Lemma 4.2 it follows that $\Phi_{t,x} \in \mathbb{D}^{1,2}(\mathcal{H}_T)$. Fix a CONS of \mathcal{H}_T that we denote by $(e_k)_{k \in \mathbb{N}}$. As has been pointed out in Section 4, the real-valued process $D^{e_k} \Phi_{t,x} = \langle D\Phi_{t,x}, e_k \rangle_{\mathcal{H}_T}$ belongs to \mathcal{P}_0 . Moreover, by the results of that section, it also belongs to the domain of the divergence operator. Thus, the assumptions of [14, Proposition 1.3.2] are fulfilled and hence we have

$$\begin{aligned} D^{e_k} \left(\int_0^t \int_{\mathbb{R}^d} \Phi_{t,x}(s, z)M(ds, dz) \right) &= D^{e_k}(\delta(\Phi_{t,x})) \\ &= \langle \Phi_{t,x}, e_k \rangle_{\mathcal{H}_T} + \delta(D^{e_k} \Phi_{t,x}), \end{aligned}$$

for any $k \in \mathbb{N}$. This proves (5.1). \square

Our next aim is to prove a result on commutation of the Malliavin derivative operator with the pathwise integral (2.20). In [8, Lemma 2.2] a similar question is analyzed. However, that version seems not to be directly applicable to our context.

Proposition 5.2 *Let Λ fullfil **(A6)** and either **(A4)** or **(A7)**. Let Z be a stochastic process satisfying the same assumptions as in Proposition 5.1. Then for all $(t, x) \in [0, T] \times \mathbb{R}^d$,*

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)Z(s, z)dzds \in \mathbb{D}^{1,2}$$

and

$$D \left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)Z(s, z)dzds \right) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)DZ(s, z)dzds. \quad (5.2)$$

Proof. Let $\Lambda = 1_{(t-a, t-b]} 1_{\{x\} - A}$ for some $0 \leq a < b \leq t$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$ where $\{x\} - A = \{x - z, z \in A\}$. In this case, formula (5.2) reads

$$D \left(\int_a^b \int_A Z(s, z)dzds \right) = \int_a^b \int_A DZ(s, z)dzds \quad (5.3)$$

almost surely. This follows from the arguments in the above-mentioned reference [8]. We notice that a direct proof of (5.3) can also be done using the definition of the Malliavin operator as a directional derivative.

In the next step, we consider $\Lambda \in L^2([0, T]; L^1(\mathbb{R}^d))$. We recall that according to [3, (3.13)], the pathwise integral is almost surely well defined as a pathwise Lebesgue integral. Linear combinations of products of indicator functions as those considered

in the previous step, are dense in $L^2([0, T]; L^1(\mathbb{R}^d))$. Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of such simple functions converging to Λ in $L^2([0, T]; L^1(\mathbb{R}^d))$. Then,

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} (\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)) Z(s, z) dz ds \right)^2 \right] \\ & \leq T \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)) \right. \\ & \quad \times (\Lambda(t-s, x-y) - \Lambda_n(t-s, x-y)) Z(s, z) Z(s, y) dy dz ds \left. \right] \\ & \leq C \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E}(|Z(t, x)|^2) \\ & \quad \times \int_0^t \left(\int_{\mathbb{R}^d} |\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)| dz \right)^2 ds, \end{aligned}$$

which goes to zero as $n \rightarrow \infty$.

Using similar arguments,

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) DZ(s, z) dz ds \right. \right. \\ & \quad \left. \left. - D \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) Z(s, z) sz ds \right)^2 \right] \\ & = \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} (\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)) DZ(s, z) dz ds \right)^2 \right] \\ & \leq C \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [\|DZ(t, x)\|_{\mathcal{H}_T}^2] \\ & \quad \times \int_0^t \left(\int_{\mathbb{R}^d} |\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)| dz \right)^2 ds, \end{aligned}$$

where in the first equality we have used the first step of this proof. The last term goes to zero as $n \rightarrow \infty$. Since D is a closed operator, the Proposition holds for $\Lambda \in L^2([0, T]; L^1(\mathbb{R}^d))$.

Finally, assume that Λ satisfies the assumptions of the Proposition. Let $\Lambda_n \in L^2([0, T], \mathcal{S}(\mathbb{R}^d))$, $n \in \mathbb{N}$ be as in 2.1. Then, according to [3] (see also Section 2) we have,

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} (\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)) Z(s, z) dz ds \right)^2 \right] \\ & = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\eta)|^2 |\mathcal{F}\psi_n(\eta) - 1|^2 \nu_s^Z(d\eta) ds. \end{aligned}$$

This goes to zero as $n \rightarrow \infty$, by dominated convergence.

Similarly,

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) DZ(s, z) dz ds \right. \right. \\
& \quad \left. \left. - D \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) Z(s, z) dz ds \right)^2 \right] \\
& = \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} (\Lambda(t-s, x-z) - \Lambda_n(t-s, x-z)) DZ(s, z) dz ds \right)^2 \right] \\
& = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\eta)|^2 |\mathcal{F}\psi_n(\eta) - 1|^2 \nu_s^{DZ}(d\eta) ds,
\end{aligned}$$

converges to zero as $n \rightarrow \infty$. Notice that by Theorem 3.2, the integrals involved in these computations exist. By the closedness of the Malliavin derivative operator, we conclude the proof. \square

6 Malliavin differentiability of the solution of the SPDE

This section is devoted to prove that the solution to the stochastic partial differential equation (1.2) at a given point $(t, x) \in [0, T] \times \mathbb{R}^d$ is differentiable in Malliavin's sense. We also derive an SPDE satisfied by the \mathcal{H}_T -valued stochastic process $\{Du(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$. This general result applies in particular to the solution of the stochastic wave equation in any spatial dimension.

It is assumed that Λ satisfies **(A3)**, **(A6)** and either **(A4)** or **(A5)** and **(A7)**. For its further use, we introduce an SPDE more general than (1.2), as follows. Let $h \in \mathcal{H}_T$ and consider

$$\begin{aligned}
u^h(t, x) &= \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \sigma(u(s, z)) M(ds, dz) \\
&+ \int_0^t \langle \Lambda(t-s, x-*) \sigma(u^h(s, *)), h(s) \rangle_{\mathcal{H}} ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) b(u(s, z)) dz ds. \tag{6.1}
\end{aligned}$$

It is easy to check that the Picard iterations $\{u^{m,h}(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$, $m \in \mathbb{N}$, satisfy the S property of [3, Lemma 4.5]. With this, an easy extension of [3, Theorem 4.2, Theorem 4.8] provides existence (and uniqueness) of a random field solution $\{u^h(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ to (6.1). Moreover,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \sup_{\|h\|_{\mathcal{H}_T} \leq c} E \left[|u^h(t, x)|^2 \right] < \infty. \tag{6.2}$$

The details of the proof are left to the reader.

Owing to the results proved in section 4, the stochastic integral in (6.1) can be interpreted either as a Conus-Dalang's integral, a Skorohod integral, or as

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \sigma(u(s, z)) M(ds, dz) := \sum_{k \in \mathbb{N}} \int_0^t \langle \Lambda(t-s, x-*) \sigma(u^h(s, *)), e_k(s) \rangle_{\mathcal{H}} dW_s^k,$$

with $(e_k)_{k \in \mathbb{N}}$ a CONS of \mathcal{H}_T and $(W_s^k, t \in [0, T])_{k \in \mathbb{N}}$ a sequence of independent standard Brownian motions.

Similarly

$$\int_0^t \langle \Lambda(t-s, x-*) \sigma(u^h(s, *)), h(s) \rangle_{\mathcal{H}} ds = \sum_{k \in \mathbb{N}} \int_0^t \langle \Lambda(t-s, x-*) \sigma(u^h(s, *)), e_k(s) \rangle_{\mathcal{H}} h^k(s) ds,$$

where $h^k(s) = \langle h(s), e_k(s) \rangle_{\mathcal{H}}$, $k \in \mathbb{N}$.

Throughout the section, we shall use the isometry between the spaces \mathcal{H}_T and \mathbb{H} defined in Section 2 and the abstract Wiener space $(\Omega, \mathbb{H}, \mathbb{P})$ described in that section.

The objective is to prove the following.

Theorem 6.1 *We assume that Λ satisfies the assumptions **(A3)**, **(A6)** and either **(A4)** or **(A5)** and **(A7)**. We also suppose that the coefficients σ and b are continuously differentiable real-valued functions with bounded derivatives. Then for any $(t, x) \in [0, T] \times \mathbb{R}^d$, $u(t, x) \in \mathbb{D}^{1,2}$. Moreover, the stochastic process $\{Du(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ satisfies the SPDE*

$$\begin{aligned} Du(t, x) = & \Lambda(t - \cdot, x - *) \sigma(u(\cdot, *)) \\ & + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \sigma'(u(s, z)) Du(s, z) M(ds, dz) \\ & + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) b'(u(s, z)) Du(s, z) ds dz, \end{aligned} \quad (6.3)$$

where $\Lambda(t - \cdot, x - *) \sigma(u(\cdot, *))$ is the stochastic process derived in Lemma 4.1 section 3, for $Z := \sigma(u)$.

The proof of Theorem 6.1 will be carried out in two steps. Firstly, we will show that $u(t, x) \in \mathbb{D}^{1,2}$ and in a second step, we shall establish (6.3). The proof of the former statement relies on [[14] Lemma 1.2.3]. For the sake of completeness, we quote this result.

Lemma 6.2 *Let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{D}^{1,2}$ such that $\lim_{n \rightarrow \infty} F_n = F$ in $L^2(\Omega)$ and $\sup_{n \in \mathbb{N}} \mathbb{E}[\|DF_n\|_{\mathcal{H}_T}^2] < \infty$. Then $F \in \mathbb{D}^{1,2}$ and the sequence $(DF_n)_{n \in \mathbb{N}}$ converges to DF in the weak topology of $L^2(\Omega; \mathcal{H}_T)$.*

This Lemma will be applied to the sequence $F_n := u_n(t, x)$, $n \in \mathbb{N}$, where $(t, x) \in [0, T] \times \mathbb{R}^d$ is fixed, and $u_n(t, x)$ is given by the solution to the evolution equation

$$\begin{aligned} u_n(t, x) &= \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \sigma(u_n(s, z)) M(ds, dz) \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) b(u_n(s, z)) dz ds, \end{aligned} \quad (6.4)$$

with Λ_n defined in (2.1).

Assume that the functions σ, b are Lipschitz continuous. Since $\Lambda_n \in \mathcal{S}(\mathbb{R}^d)$, the stochastic integral in (6.4) is a Walsh's integral (see [24]). It is well-known that (6.4) has a unique random field solution, and that it satisfies the S -property. In particular for each $n \in \mathbb{N}$, the process $\{Z(t, x) := u_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ satisfies the assumptions **(A1)**, **(A2)**. For a proof of these results, we can proceed as in [5, Theorem 13].

From the proof of Proposition 7.1 in [20] we obtain the following.

Proposition 6.3 *Let Λ_n be as in (2.1). Assume that the coefficients σ, b in (6.4) are continuously differentiable with bounded derivatives. Then for each $n \in \mathbb{N}$ and every $(t, x) \in [0, T] \times \mathbb{R}^d$, the random variable $u_n(t, x)$ belongs to $\mathbb{D}^{1,2}$. Moreover, the \mathcal{H}_T -valued stochastic process $\{Du_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ is the solution to the SPDE*

$$\begin{aligned} Du_n(t, x) &= \Lambda_n(t - \cdot, x - *) \sigma(u_n(\cdot, *)) \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \sigma'(u_n(s, z)) Du_n(s, z) M(ds, dz) \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) b'(u_n(s, z)) Du_n(s, z) dz ds. \end{aligned} \quad (6.5)$$

Next, we study the convergence of the sequence of processes $(u_n)_{n \in \mathbb{N}}$ to u .

Proposition 6.4 *We assume that Λ satisfies the hypotheses of Theorem 6.1. Moreover, we suppose that the functions σ and b are Lipschitz continuous. Then we have*

$$\lim_{n \rightarrow \infty} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E}[|u_n(t, x) - u(t, x)|^2] = 0.$$

Proof. We start by proving that

$$\sup_{n \in \mathbb{N}} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E}[|u_n(t, x)|^2] < \infty. \quad (6.6)$$

Indeed, from (6.4) it follows that $\mathbb{E}[|u_n(t, x)|^2] \leq 2(I_{1,n}(t, x) + I_{2,n}(t, x))$, for every $(t, x) \in [0, T] \times \mathbb{R}^d$, where

$$I_{1,n}(t, x) = \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \sigma(u_n(s, z)) M(ds, dz) \right)^2 \right]$$

and

$$I_{2,n}(t, x) = \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) b(u_n(s, z)) dz ds \right)^2 \right].$$

Notice that the inequalities (2.17), (2.21) also hold with Λ replaced by $\Lambda_n(t-\cdot, x-\cdot)$. Then, by taking $Z(t, x) := \sigma(u_n(t, x))$ and $Z(t, x) := b(u_n(t, x))$, respectively, we obtain

$$\begin{aligned} I_{1,n}(t, x) &\leq \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[\sigma(u_n(r, y))^2] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi + \eta)|^2 \mu(d\xi) ds \\ &\leq C \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[(1 + u_n(r, y))^2] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi + \eta)|^2 \mu(d\xi) ds \\ &\leq C \int_0^t \left(1 + \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[u_n(r, y)^2] \right) J_1(t-s) ds, \end{aligned}$$

and

$$\begin{aligned} I_{2,n}(t, x) &\leq \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[b(u_n(r, y))^2] \sup_{\eta \in \mathbb{R}} |\mathcal{F}\Lambda_n(t-s)(\eta)|^2 ds \\ &\leq C \int_0^t \left(1 + \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[u_n(r, y)^2] \right) J_2(t-s) ds, \end{aligned}$$

where the functions J_1 and J_2 are defined in (2.22) and (2.23), respectively. This yields

$$\begin{aligned} \sup_{(r,y) \in [0,t] \times \mathbb{R}^d} \mathbb{E}[|u_n(r, y)|^2] &\leq C \int_0^t \left(1 + \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[u_n(r, y)^2] \right) \\ &\quad \times (J_1(t-s) + J_2(t-s)) ds. \end{aligned}$$

Using the version of Gronwall's Lemma in [5, Lemma 15] along with (2.24) yields (6.6).

Next, we show the assertion of the proposition. Using equations (1.2) and (6.4), we have

$$\mathbb{E}[|u(t, x) - u_n(t, x)|^2] \leq C(T_{1,n}(t, x) + T_{2,n}(t, x) + T_{3,n}(t, x) + T_{4,n}(t, x)),$$

where

$$\begin{aligned} T_{1,n}(t, x) &= \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \right. \right. \\ &\quad \times \left. \left. (\sigma(u_n(s, z)) - \sigma(u(s, z))) M(ds, dz) \right)^2 \right], \end{aligned}$$

$$\begin{aligned}
T_{2,n}(t, x) &= \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} (\Lambda_n(t-s, x-z) - \Lambda(t-s, x-z)) \right. \right. \\
&\quad \times \sigma(u(s, z)) M(ds, dz) \left. \right)^2 \left. \right], \\
T_{3,n}(t, x) &= \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) (b(u_n(s, z)) - b(u(s, z))) dz ds \right)^2 \right], \\
T_{4,n}(t, x) &= \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} (\Lambda_n(t-s, x-z) - \Lambda(t-s, x-z)) \right. \right. \\
&\quad \times b(u(s, z)) dz ds \left. \right)^2 \left. \right].
\end{aligned}$$

For the terms $T_{1,n}(t, x)$, $T_{2,n}(t, x)$, we apply the inequality (2.17) in the following situations. For the former term, we replace Λ by $\Lambda_n(t - \cdot, x - *)$ and take $Z := \sigma(u_n) - \sigma(u)$; for the latter, we replace Λ by $[\Lambda_n - \Lambda](t - \cdot, x - *)$ and take $Z := \sigma(u)$. This yields

$$\begin{aligned}
T_{1,n}(t, x) &\leq \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} [|\sigma(u_n(r, y)) - \sigma(u(r, y))|^2] \\
&\quad \times \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi + \eta)|^2 \mu(d\xi) ds \\
&\leq C \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} [|u_n(r, y) - u(r, y)|^2] J_1(t-s) ds,
\end{aligned}$$

and

$$\begin{aligned}
T_{2,n}(t, x) &= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi) - \mathcal{F}\Lambda(t-s)(\xi)|^2 \mu_s^{\sigma(u)}(d\xi) ds \\
&= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\xi)|^2 |\mathcal{F}\psi_n(\xi) - 1|^2 \mu_s^{\sigma(u)}(d\xi) ds.
\end{aligned}$$

For the term $T_{3,n}(t, x)$, we apply (2.21) with Λ replaced by $\Lambda_n(t - \cdot, x - *)$ and $Z := b(u_n) - b(u)$. For $T_{4,n}(t, x)$, we proceed similarly with Λ replaced by $[\Lambda_n - \Lambda](t - \cdot, x - *)$ and $Z := b(u)$, respectively. We obtain

$$\begin{aligned}
T_{3,n}(t, x) &\leq \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} [|b(u_n(r, y)) - b(u(r, y))|^2] \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\eta)|^2 ds \\
&\leq C \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E} [|u_n(r, y) - u(r, y)|^2] J_2(t-s) ds, \\
T_{4,n}(t, x) &= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\eta) - \mathcal{F}\Lambda(t-s)(\eta)|^2 \nu_s^{\sigma(u)}(d\eta) ds
\end{aligned}$$

$$= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\eta)|^2 |\mathcal{F}\psi_n(\eta) - 1|^2 \nu_s^{\sigma(u)}(d\eta) ds.$$

The terms $T_{2,n}(t, x)$, $T_{4,n}(t, x)$ converge to zero as $n \rightarrow \infty$ uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$, by dominated convergence. Hence, altogether we have

$$\begin{aligned} & \sup_{(r,y) \in [0,t] \times \mathbb{R}^d} \mathbb{E}[|u(r,y) - u_n(r,y)|^2] \\ & \leq C_n + C \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[|u(r,y) - u_n(r,y)|^2] (J_1(t-s) + J_2(t-s)) ds, \end{aligned}$$

where C_n tends to 0 as $n \rightarrow \infty$ uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$. An application of Gronwall's Lemma yields the assertion. \square

The next proposition provides the last ingredient for the application of Lemma 6.2.

Proposition 6.5 *With the same assumptions as in Theorem 6.1, we have*

$$\sup_{n \in \mathbb{N}} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[\|Du_n(t,x)\|_{\mathcal{H}_T}^2] < \infty.$$

Proof. We bound the $L^2(\Omega; \mathcal{H}_T)$ -norm of each term on the right-hand side of (6.5). For the first term, we apply (4.1) with $Z := \sigma(u_n)$ and then, (2.14), (2.17). By the properties of σ , we obtain

$$\begin{aligned} & \mathbb{E}[\|\Lambda_n(t - \cdot, x - *)\sigma(u_n(\cdot, *))\|_{\mathcal{H}_T}^2] \\ & = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi)|^2 \mu_s^{\sigma(u_n)}(d\xi) ds \\ & \leq \int_0^t \mathbb{E}[\sigma(u_n(s, 0))^2] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi + \eta)|^2 \mu(d\xi) ds \\ & \leq C \left(1 + \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[|u_n(r,y)|^2] \right) \int_0^t \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\xi + \eta)|^2 \mu(d\xi) ds. \end{aligned}$$

The last term is uniformly bounded in $n \in \mathbb{N}$ and $(t, x) \in [0, T] \times \mathbb{R}^d$, due to (6.6) and assumption **(A3)**.

For the second term on the right-hand side of (6.5), we apply (3.1) with Λ replaced by $\Lambda_n(t - \cdot, x - *)$, $Z := \sigma'(u_n)Du_n$ and $\mathcal{A} = \mathcal{H}_T$. Since σ' is bounded, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \sigma'(u_n(s, z)) Du_n(s, z) M(ds, dz) \right\|_{\mathcal{H}_T}^2 \right] \\ & \leq \int_0^t \mathbb{E}[\|\sigma'(u_n(s, 0))Du_n(s, 0)\|_{\mathcal{H}_T}^2] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\xi + \eta)|^2 \mu(d\xi) ds \end{aligned}$$

$$\leq C \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[\|Du_n(r,y)\|_{\mathcal{H}_T}^2] J_1(t-s) ds.$$

Finally, applying (3.2) with Λ replaced by $\Lambda_n(t - \cdot, x - *)$, $Z := b'(u_n)Du_n$ and $\mathcal{A} = \mathcal{H}_T$ yields

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) b'(u_n(s,z)) Du_n(s,z) dz ds \right\|_{\mathcal{H}_T}^2 \right] \\ & \leq \int_0^t \mathbb{E}[\|b'(u_n(s,0))Du_n(s,0)\|_{\mathcal{H}_T}^2] \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda_n(t-s)(\eta)|^2 ds \\ & \leq C \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[\|Du_n(r,y)\|_{\mathcal{H}_T}^2] J_2(t-s) ds. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathbb{E}[\|Du_n(t,x)\|_{\mathcal{H}_T}^2] \\ & \leq C \left[1 + \int_0^t \sup_{(r,y) \in [0,s] \times \mathbb{R}^d} \mathbb{E}[\|Du_n(r,y)\|_{\mathcal{H}_T}^2] (J_1(t-s) + J_2(t-s)) ds \right]. \end{aligned}$$

An application of Gronwall's Lemma finishes the proof. \square

Propositions 6.4, 6.5, along with Lemma 6.2 yields that $u(t,x) \in \mathbb{D}^{1,2}$ for any $(t,x) \in [0,T] \times \mathbb{R}^d$. This is the first assertion of Theorem 6.1.

The rest of this section is devoted to prove that the Malliavin derivative of the process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfies (6.3). For this, we consider the equation (1.2) satisfied by this process and apply the Malliavin derivative operator to each term. We obtain

$$\begin{aligned} Du(t,x) &= D \left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \sigma(u(s,z)) M(ds, dz) \right) \\ &+ D \left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) b(u(s,z)) dz ds \right). \end{aligned} \tag{6.7}$$

Then (6.3) will follow by applying Propositions 5.1, 5.2. The rest of this section is devoted to check that the stochastic processes $Z(t,x) := \sigma(u(t,x))$ and $Z(t,x) := b(u(t,x))$, $(t,x) \in [0,T] \times \mathbb{R}^d$, satisfy the assumptions of these propositions, respectively.

Lemma 6.6 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function. Then the stochastic process $g(u) = \{g(u(t,x)), (t,x) \in [0,T] \times \mathbb{R}^d\}$, where $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is the solution of (1.2), satisfies the assumptions **(A1)**, **(A2)**.*

Proof. Since the process u is predictable and g is continuous, $g(u)$ is clearly predictable. The function g has linear growth; along with (6.2), this yields

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [g(u(t,x))^2] \leq C \left[1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} (|u(t,x)|^2) \right] < \infty.$$

The proof of **(A2)** follows from the S -property of the process u (see [3, Definition 4.4, Lemma 4.5 and Theorem 4.2]). \square

Lemma 6.7 *Let $g(u) = \{g(u(t,x)), (t,x) \in [0,T] \times \mathbb{R}^d\}$ be as in Lemma 6.6. Assume in addition that g is continuous differentiable with bounded derivative. Then the \mathcal{H}_T -valued stochastic process $D(g(u)) : \{D(g(u(t,x))), (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfies **(A8)**, **(A9)**.*

Proof. First, we note that by the construction of the Malliavin derivative based on smooth functionals (see for instance [14, (1.29)]), the stochastic process $D(g(u))$ inherits the predictability property of the process u . We also notice that by the chain rule of Malliavin calculus, $g(u(t,x)) \in \mathbb{D}^{1,2}$ and $D(g(u(t,x))) = g'(u((t,x))Du(t,x)$, for any $(t,x) \in [0,T] \times \mathbb{R}^d$.

We are assuming that g' is bounded. Thus,

$$\begin{aligned} \mathbb{E} [\|D(g(u(t,x)))\|_{\mathcal{H}_T}^2] &\leq C \mathbb{E} [\|Du(t,x)\|_{\mathcal{H}_T}^2] \leq C \liminf_{n \rightarrow \infty} \mathbb{E} [\|Du_n(t,x)\|_{\mathcal{H}_T}^2] \\ &\leq C \sup_{n \in \mathbb{N}} \mathbb{E} [\|Du_n(t,x)\|_{\mathcal{H}_T}^2], \end{aligned}$$

where $u_n(t,x)$ is defined by (6.4). In the second inequality above, we have used that the sequence $(Du_n(t,x))_{n \in \mathbb{N}}$ converges weakly in \mathcal{H}_T to $Du(t,x)$ along with [9, Theorem 5, Chapter 10]. From Proposition 6.5, we conclude

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [\|D(g(u(t,x)))\|_{\mathcal{H}_T}^2] < \infty.$$

Hence the stochastic process $D(g(u))$ satisfies **(A8)**.

Consider the Picard iterations of the processes $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, $\{u^h(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, $h \in \mathcal{H}_T$, that we denote by $\{u^m(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, $\{u^{m,h}(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, $m \geq 1$, respectively. We have the following:

(SP) for any $m \geq 1$, the process

$$(u^m(t,x), u^{m,h}(t,x), u^{m-1}(t,x), u^{m-1,h}(t,x), (t,x) \in [0,T] \times \mathbb{R}^d),$$

satisfies the S -property defined in [3, Definition 4.4].

Indeed, this can be proved by a recursive argument on $m \geq 1$, following similar arguments as in [3, Lemma 4.5].

Property **(SP)** implies that the process $D(g(u))$ satisfies **(A9)**. Indeed, Let $(e_k)_{k \in \mathbb{N}}$ be a CONS of \mathcal{H}_T . The Malliavin derivative $D^{e_k}u(t, x)$ can be obtained as

$$L^2(\Omega) - \lim_{\epsilon \rightarrow 0} \frac{u^{\epsilon e_k}(t, x) - u(t, x)}{\epsilon}.$$

Then, using the chain rule of Malliavin calculus, we conclude

$$\begin{aligned} & \mathbb{E}[D^{e_k}(g(u(t, x)))D^{e_k}(g(u(t, x + y)))] \\ &= L^2(\Omega) - \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[g'(u(t, x)) \frac{u^{\epsilon e_k}(t, x) - u(t, x)}{\epsilon} \right. \\ & \quad \left. \times g'(u(t, x + y)) \frac{u^{\epsilon e_k}(t, x + y) - u(t, x + y)}{\epsilon} \right] \end{aligned} \quad (6.8)$$

$$= \mathbb{E}[D^{e_k}(g(u(t, 0)))D^{e_k}(g(u(t, y)))], \quad (6.9)$$

where the last equality is a consequence of **(SP)**. \square

7 Existence of density

In this section we consider the solution to the SPDE (1.2) at a fixed point $(t, x) \in]0, T] \times \mathbb{R}^d$ in the particular case where σ is constant. Under suitable assumptions, we prove that the law of $u(t, x)$ has a density with respect to the Lebesgue measure on \mathbb{R} .

Theorem 7.1 *We assume that Λ satisfies the same assumptions than in Theorem 6.1 and that*

$$g(\delta) := \int_0^\delta \|\Lambda(s, *)\|_{\mathcal{H}}^2 ds > 0,$$

for any $\delta > 0$. Suppose that the function σ is constant, and b is continuously differentiable with bounded derivative. Then, for any $(t, x) \in]0, T] \times \mathbb{R}^d$ the probability law of $u(t, x)$ has a density.

Proof. We apply Bouleau-Hirsch's criterion, see e.g. [14, Section 2.13]. Fix $(t, x) \in]0, T] \times \mathbb{R}^d$. We already know from Theorem 6.1 that $u(t, x) \in \mathbb{D}^{1,2}$. Thus, it suffices to show that

$$\|Du(t, x)\|_{\mathcal{H}_T}^2 > 0, \text{ a.s.} \quad (7.1)$$

From (6.3), and for $\delta \in]0, t]$, we obtain

$$\begin{aligned} \|Du(t, x)\|_{\mathcal{H}_T}^2 &= \int_0^t \|D_{s,*}u(t, x)\|_{\mathcal{H}}^2 ds \geq \int_{t-\delta}^t \|D_{s,*}u(t, x)\|_{\mathcal{H}}^2 ds \\ &\geq \frac{1}{2}\sigma^2 \int_{t-\delta}^t \|\Lambda(t-s, x-*)\|_{\mathcal{H}}^2 ds - I(t, x; \delta), \end{aligned} \quad (7.2)$$

where

$$I(t, x; \delta) = \int_{t-\delta}^\delta ds \left\| \int_0^t dr \int_{\mathbb{R}^d} dz \Lambda(t-r, x-z) b'(u(r, z)) D_{s,*}u(r, z) \right\|_{\mathcal{H}}^2.$$

By a change of variable, we have

$$\int_{t-\delta}^t \|\Lambda(t-s, x-*)\|_{\mathcal{H}}^2 ds = \int_0^\delta \|\Lambda(s, *)\|_{\mathcal{H}}^2 ds = g(\delta). \quad (7.3)$$

Assumption **(A3)** implies that $g(T) < \infty$. Hence,

$$\lim_{\delta \rightarrow 0} g(\delta) = 0. \quad (7.4)$$

The Malliavin derivative $D_{s,*}u(r, z)$ vanishes except if $0 \leq s \leq r$. Using this property and the change of variables $s \mapsto t-s$, $r \mapsto t-r$, we obtain

$$\mathbb{E}[I(t, x; \delta)] = \mathbb{E} \left[\int_{t-\delta}^\delta ds \left\| \int_s^t dr \int_{\mathbb{R}^d} dz \Lambda(t-r, x-z) b'(u(r, z)) D_{s,*}u(r, z) \right\|_{\mathcal{H}}^2 \right]$$

$$= \mathbb{E} \left[\int_0^\delta ds \left\| \int_0^s dr \int_{\mathbb{R}^d} dz \Lambda(r, x-z) b'(u(t-r, z)) D_{t-s,*} u(t-r, z) \right\|_{\mathcal{H}}^2 \right].$$

We apply Fubini's theorem and then, (3.2) with $\mathcal{A} := \mathcal{H}$ and $Z(r, z) := b'(u(t-r, z)) D_{t-s,*} u(t-r, z)$. Since the function b' is bounded, we obtain

$$\begin{aligned} \mathbb{E}[I(t, x; \delta)] &\leq C \int_0^\delta ds \left[\int_0^s dr \mathbb{E} \left[\|b'(u(t-r, 0)) D_{t-s,*} u(t-r, 0)\|_{\mathcal{H}}^2 \right] \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(\eta)|^2 \right] \\ &\leq C \mathbb{E} \left[\int_0^\delta ds \int_0^\delta dr \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(r)(\eta)|^2 \|D_{t-s,*} u(t-r, 0)\|_{\mathcal{H}}^2 \right] \\ &\leq C \int_0^\delta dr \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(r)(\eta)|^2 \mathbb{E} \left[\|D_{t-s,*} u(t-r, 0)\|_{\mathcal{H}_\delta}^2 \right]. \end{aligned} \quad (7.5)$$

The next objective is to prove that

$$\sup_{0 \leq r \leq \delta} \mathbb{E} \left[\|D_{t-s,*} u(t-r, 0)\|_{\mathcal{H}_\delta}^2 \right] \leq C g(\delta). \quad (7.6)$$

Indeed, owing to (6.3), and by applying once more (3.2), we have

$$\begin{aligned} &\mathbb{E} \left[\|D_{t-s,*} u(t-r, 0)\|_{\mathcal{H}_\delta}^2 \right] \\ &\leq 2\sigma^2 g(\delta) + 2\mathbb{E} \left[\left\| \int_0^{t-r} ds \int_{\mathbb{R}^d} dy \Lambda(t-r-s, z-y) b'(u(s, y)) D_{t-s,*} u(s, y) \right\|_{\mathcal{H}_\delta}^2 \right] \\ &\leq C_1 g(\delta) + C_2 \int_0^{t-r} ds \left(\sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 \right) \mathbb{E} \left[\|D_{t-s,*} u(s, 0)\|_{\mathcal{H}_\delta}^2 \right]. \end{aligned}$$

Hence, (7.6) follows from an application of a version of Gronwall's Lemma.

From (7.5), (7.6), we obtain

$$\mathbb{E}[I(t, x; \delta)] \leq C g(\delta) \bar{g}(\delta), \quad (7.7)$$

with

$$\bar{g}(\delta) := \int_0^\delta ds \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2.$$

Notice that, assumption **(A6)** implies

$$\lim_{\delta \rightarrow 0} \bar{g}(\delta) = 0.$$

Fix $\delta \in]0, t[$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\frac{1}{n} < \frac{\sigma^2}{3} g(\delta)$. Using Chebyshev's inequality along with (7.2), (7.3), (7.7) yield

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\|Du(t, x)\|_{\mathcal{H}_T}^2 < \frac{1}{n} \right] \leq \lim_{n \rightarrow \infty} \mathbb{P} \left[I(t, x; \delta) \geq \frac{\sigma^2}{2} g(\delta) - \frac{1}{n} \right]$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \left(\frac{\sigma^2}{2} g(\delta) - \frac{1}{n} \right)^{-1} \mathbb{E} [I(t, x; \delta)] \\
&\leq C \bar{g}(\delta).
\end{aligned}$$

Letting $\delta \rightarrow 0$, we obtain

$$\mathbb{P} [\|Du(t, x)\|_{\mathcal{H}_T}^2 = 0] = 0.$$

This is equivalent to (7.1). \square

Consider the particular case of the stochastic wave equation in spatial dimension $d > 3$. The Fourier transform of the fundamental solution of the corresponding partial differential equation is given by

$$\mathcal{F}\Lambda(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Hence, there exist constants C_1, C_2 , depending on T , such that for any $t \in [0, T]$, and $\xi \in \mathbb{R}^d$,

$$\frac{C_1}{1 + |\xi|^2} \leq \frac{\sin^2(2\pi t|\xi|)}{4\pi^2|\xi|^2} \leq \frac{C_2}{1 + |\xi|^2}. \quad (7.8)$$

Assume that the spectral measure μ satisfies

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi + \eta|^2} < \infty. \quad (7.9)$$

Then, according to [3, Theorem 5.1], Λ satisfies the conditions **(A3)**, **(A4)** and **(A6)**. Hence, Theorem 6.1 holds.

Property (7.9) along with (7.8) imply

$$g(\delta) \geq C_1 \int_0^\delta ds \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} \geq C\delta.$$

This yields the following result

Theorem 7.2 *Consider the particular case where Λ is the fundamental solution of the wave equation with $d \in \mathbb{N}$. Assume (7.9) and that the functions σ and b are as in Theorem 7.1. Then, the statement of this theorem holds.*

Assume that the covariance measure Γ has a density: $\Gamma(dx) = f(x)dx$, with $f \geq 0$. In [17], it is proved that (7.9) is equivalent to $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty$. This condition is satisfied for example if $f(x) = |x|^{-\beta}$, $\beta \in]0, 2[$, a case that has been extensively studied in the literature of SPDEs driven by correlated noises.

Remark

As has been already mentioned in the introduction, so far the existence of density for the probability law of the solution of a SPDE like (1.2) has been established when Λ is a non-negative distribution. In this case, it is proved that the dominant term in the analysis of the Malliavin matrix is the first term on the right-hand side of (6.3). Assuming that the coefficient $|\sigma| \geq \sigma_0 > 0$, we have

$$\|\Lambda(t - \cdot, x - *)\sigma(u(\cdot, *))\|_{\mathcal{H}_T}^2 \geq \sigma_0^2 \|\Lambda(t - \cdot, x - *)\|_{\mathcal{H}_T}^2. \quad (7.10)$$

Then, the result is obtained by following a similar argument as in the proof of Theorem 7.1.

For the wave operator in spatial dimension $d > 3$, Λ fails to satisfy the non-negativeness requirement. So far, we have not been able to have a suitable lower bound like for instance in (7.10). We notice that the trivial lower bound

$$\begin{aligned} \|\Lambda(t - \cdot, x - *)\sigma(u(\cdot, *))\|_{\mathcal{H}_T}^2 &= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t - s)(\xi)|^2 \mu_s^{\sigma(u)} d(\xi) ds \\ &\geq \sigma_0^2 \int_0^t \inf_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(t - s)(\xi + \eta)|^2 \mu_s^{\sigma(u)} d(\xi) ds, \end{aligned}$$

does not help. Indeed, if μ is the spectral measure of a Riesz kernel ($\Gamma(dx) = |x|^{-\beta} dx$, $\beta \in]0, 2[$), one can prove that the last integral in the above expression vanishes.

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